Math 43 Midterm 2 Review Answers

[1] [a] $x(1-t)=t \rightarrow x-xt=t \rightarrow x=t+xt \rightarrow x=t(1+x) \rightarrow t=\frac{x}{1+x}$

$y=\frac{\frac{x}{1+x}-1}{1+\frac{x}{1+x}} \rightarrow y=\frac{x-(1+x)}{1+x+x} \rightarrow \boxed{y=\frac{2x-1}{2x+1}}$

[b] $\tan t=\frac{x-3}{5}$ and $\sec t=\frac{y-4}{2}$ and $\sec^2 t-\tan^2 t=1 \rightarrow \boxed{\frac{(y-4)^2}{4}-\frac{(x-3)^2}{25}=1}$

[c] $\cos t=\frac{x-8}{6}$ and $\sin t=7-y$ and $\cos^2 t+\sin^2 t=1 \rightarrow \frac{(x-8)^2}{36}+(7-y)^2=1$

$\rightarrow \boxed{\frac{(x-8)^2}{36}+(y-7)^2=1}$

[d] $\frac{x}{5}=\ln 4t \rightarrow e^{\frac{x}{5}}=4t \rightarrow t=\frac{1}{4}e^{\frac{x}{5}} \rightarrow y=2\left(\frac{1}{4}e^{\frac{x}{5}}\right)^3 \rightarrow \boxed{y=\frac{1}{32}e^{\frac{3x}{5}}}$

[2] [a] $x=-3+(7-(-3))t$
$y=-6+(-2-(-6))t$ $\rightarrow$ $\boxed{\begin{aligned}x&=-3+10t\\ y&=-6+4t\end{aligned}}$

[b] $\text{center}=\left(\frac{-3+7}{2},\frac{-6+(-2)}{2}\right)=(2,-4)$ $\quad$ $\text{radius}=\frac{1}{2}\sqrt{(7-(-3))^2+(-2-(-6))^2}=\frac{\sqrt{116}}{2}=\sqrt{29}$

$\boxed{\begin{aligned}x&=2+\sqrt{29}\cos t\\ y&=-4+\sqrt{29}\sin t\end{aligned}}$

[c] $\text{center}=\left(\frac{-3+7}{2},-6\right)=(2,-6) \rightarrow c=7-2=5$ and $b=-2-(-6)=4$

(horizontal major axis) (vertical minor axis)

$a^2=4^2+5^2=41 \rightarrow a=\sqrt{41}$

$\boxed{\begin{aligned}x&=2+\sqrt{41}\cos t\\ y&=-6+4\sin t\end{aligned}}$

[d] $\text{center}=\left(\frac{-3+7}{2},-6\right)=(2,-6) \rightarrow c=2-(-5)=7$ and $a=7-2=5$

(horizontal transverse axis)

$b^2=7^2-5^2=24 \rightarrow b=2\sqrt{6}$

$\boxed{\begin{aligned}x&=2+5\sec t\\ y&=-6+2\sqrt{6}\tan t\end{aligned}}$

[e] $\boxed{\begin{aligned}&x=t\\ &y=2t^4-3t^3+1\\ &t\in[-1,2]\end{aligned}}$

[3] center = $(70, 0)$, radius = 50, period = 20

$$x = 70 + 50\cos\left(\frac{2\pi}{20}\right)t \qquad \rightarrow \qquad \boxed{x = 70 + 50\cos\frac{\pi t}{10}}$$

$$y = 0 + 50\sin\left(\frac{2\pi}{20}\right)t \qquad\qquad \boxed{y = 50\sin\frac{\pi t}{10}}$$

[4] $(-1)^3 3(3-4) \;+\; (-1)^4 4(4-4) \;+\; (-1)^5 5(5-4) \;+\; (-1)^6 6(6-4) \;+\; (-1)^7 7(7-4) \;+\; (-1)^8 8(8-4)$

$= 3 + 0 - 5 + 12 - 21 + 32$

$= \boxed{21}$

[5] $$\frac{200!}{4!\cdot 196!} \;=\; \frac{200\cdot 199\cdot 198\cdot 197\cdot 196!}{24\cdot 196!} \;=\; \boxed{64{,}684{,}950}$$

[6] $0.4 + 0.072 + 0.00072 + 0.0000072 + \cdots$

$$= \frac{4}{10} + \left(\frac{72}{1000} + \frac{72}{100000} + \frac{72}{10000000} + \cdots\right)$$

$$= \frac{2}{5} + \left(\frac{\frac{72}{1000}}{1 - \frac{1}{100}}\right)$$

$$= \frac{2}{5} + \left(\frac{\frac{72}{1000}}{\frac{99}{100}}\right)$$

$$= \frac{2}{5} + \frac{72}{1000}\,\frac{100}{99}$$

$$= \frac{2}{5} + \frac{4}{55}$$

$$= \boxed{\frac{26}{55}}$$

[7] NOTE: The first factors in the denominator form an arithmetic sequence, and the second factors form a geometric sequence.

$$\sum_{n=1}^{9} \frac{1}{(7 - 3(n-1))\cdot 3(2)^{n-1}} \;=\; \boxed{\sum_{n=1}^{9} \frac{1}{3(10-3n)(2)^{n-1}}}$$

NOTE: To find the upper limit of summation, either solve

$7 - 3(n-1) = -17$ or $3(2)^{n-1} = 768$

$-3(n-1) = -24$ $\qquad 2^{n-1} = 256$

$n - 1 = 8$ $\qquad n - 1 = 8$

$n = 9$ $\qquad n = 9$

[8] The general term is $\binom{11}{r}(2x^5)^{11-r}(-3x^2)^r \;=\; \binom{11}{r}2^{11-r}(-3)^r(x^5)^{11-r}(x^2)^r \;=\; \binom{11}{r}2^{11-r}(-3)^r x^{55-3r}$

$$55 - 3r = 34 \quad\rightarrow\quad r = 7 \quad\rightarrow\quad \binom{11}{7}2^{11-7}(-3)^7 \;=\; \boxed{-11{,}547{,}360}$$

[9] $4(0.97)^{2(3)-1} + 4(0.97)^{2(4)-1} + 4(0.97)^{2(5)-1} + \cdots$

$= 4(0.97)^5 + 4(0.97)^7 + 4(0.97)^9 + \cdots$

$= \dfrac{4(0.97)^5}{1-(0.97)^2}$

$\approx \boxed{58.1207}$

[10] $a_2 = 2a_1 - 3 = 2(4) - 3 = 5$

$a_3 = 2a_2 - 3 = 2(5) - 3 = 7$

$a_4 = 2a_3 - 3 = 2(7) - 3 = 11$

$a_5 = 2a_4 - 3 = 2(11) - 3 = 19$

$\boxed{4, 5, 7, 11, 19}$

The sequence is neither arithmetic nor geometric. The differences are $1, 2, 4, 8$ which are not constant.

The ratios are $\dfrac{5}{4}, \dfrac{7}{5}, \dfrac{11}{7}, \dfrac{19}{11}$ which are also not constant.

[11] [a] $1(3x)^6(-2y)^0 + 6(3x)^5(-2y)^1 + 15(3x)^4(-2y)^2 + 20(3x)^3(-2y)^3$

$+ 15(3x)^2(-2y)^4 + 6(3x)^1(-2y)^5 + 1(3x)^0(-2y)^6$

$= \boxed{729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6}$

[b] $1(\sqrt{x})^4\left(-\dfrac{2}{x}\right)^0 + 4(\sqrt{x})^3\left(-\dfrac{2}{x}\right)^1 + 6(\sqrt{x})^2\left(-\dfrac{2}{x}\right)^2 + 4(\sqrt{x})^1\left(-\dfrac{2}{x}\right)^3 + 1(\sqrt{x})^0\left(-\dfrac{2}{x}\right)^4$

$= x^2 + 4x^{\frac{3}{2}}(-2x^{-1}) + 6x(4x^{-2}) + 4x^{\frac{1}{2}}(-8x^{-3}) + 16x^{-4}$

$= \boxed{x^2 - 8x^{\frac{1}{2}} + 24x^{-1} - 32x^{-\frac{5}{2}} + 16x^{-4}}$

[12] $800(0.9)^{n-1} = 3.34 \quad\rightarrow\quad (0.9)^{n-1} = 0.004175 \quad\rightarrow\quad \ln(0.9)^{n-1} = \ln 0.004175 \quad\rightarrow$

$(n-1)\ln 0.9 = \ln 0.004175 \quad\rightarrow\quad n-1 = \dfrac{\ln 0.004175}{\ln 0.9} \quad\rightarrow\quad n = 1 + \dfrac{\ln 0.004175}{\ln 0.9} \approx 53$

EJ's car was sold for scrap in $1998 + 53 = 2051$

[13] [a] PROOF:

Basis step: $1^3 = \frac{1^2(1+1)^2}{4}$

$1 = 1$

Inductive step: Assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ for some particular but arbitrary integer $k \geq 1$

Prove $1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$$\begin{aligned}
&1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 \\
&= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

So, by mathematical induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$

[b] PROOF:

Basis step: $5! > 3^{5-1}$

$120 > 81$

Inductive step: Assume $k! > 3^{k-1}$ for some particular but arbitrary integer $k \geq 5$

Prove $(k+1)! > 3^k$

$(k+1)! = (k+1)\cdot k! > (k+1)3^k > 3 \cdot 3^k = 3^{k+1}$

Since $k \geq 5$, therefore $k+1 \geq 6 > 3$

So, by mathematical induction, $n! > 3^{n-1}$ for all integers $n \geq 5$

[14] $-73 + 7(n-1) = 529 \rightarrow 7(n-1) = 602 \rightarrow n-1 = 86 \rightarrow n = 87$

$S_{87} = \frac{87}{2}(-73 + 529) = \boxed{19{,}836}$

[15] CJ's total rent will be $\frac{24}{2}(400 + 400 + (24-1)(7)) = \$11{,}532$.

DJ's total rent will be $\frac{380(1.02^{24} - 1)}{1.02 - 1} = \$11{,}560.31$.

So, DJ will have paid $\$28.31$ more rent.